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Exact solution of linear parametrically excited system with transient frequency modulation

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To demonstrate the dangers of neglecting the transient characteristics of frequency modulation in linear parametrically excited systems a model equation with variable frequency that includes components of transient behavior is investigated. The transient components of the frequency modulation decay periodically and asymptotically with time leading to a periodic closed cycle form. The exact analytical solution of the model equation is derived for the first time in the present work by introducing a transformation that maps the original model into a system of two solvable equations. The extensive investigation of the general solution and its components demonstrates that the transient characteristics of the frequency variation cannot be ignored. In fact cases where theses characteristics can result in bounded or unbounded response of the system are presented. In general transient frequency characteristics continue to drastically affect the structure and the amplitude of the general solution long after they become indistinguishable in the frequency modulation. Numerical solutions of the model equation are also presented to further support the conclusions made in this paper.

INTRODUCTION

The investigation of parametrically excited and self-excited systems with transient characteristics has many important applications in Engineering. For example, rotating structures, like bladed disks in air engines, are driven by oscillating forces with variant frequencies of their harmonics during transient operation (see reference [1] for example). Another application of such models is the variable-mass rotor fluid systems. This problem is modeled by a parametrically excited system with transient frequency features [2–4]. Moreover, transient vibration due to blade loss can also be modeled by self-excited systems with transient frequency modulation [5]. However, most of the theoretical investigations of the previously mentioned systems focus on non-linear models. It is generally thought that the likely source of unpleasant surprises in such systems comes from modal interaction due to coupling between closely spaced modes of the nonlinear model [6]. As such approaches like the Krylov-Bogolijbov asymptotic approaches [7] or the method of harmonic balances [8, 9] are widely used to analyze such systems. Numerically, on the other hand, the modal decomposition analysis is a common approach in theoretical investigation [10].

For example Irretier and Balashov [11] and Irrtier and Leul [12] analyzed transient oscillation of weakly non-linear systems with slow variable frequency. They mainly focused

on sweeping through resonance frequency of such systems. On the other hand, Asfar [13] used the method of multiple scales to investigate the quenching of self excited vibrations. In linear systems, however, transient variation of the frequency did not grasp as much attention especially if the steady state response is only considered. Although not many linear parametrically excited model equations have know analytical solution, those that have do not demonstrate the effect of their transient characteristics on the magnitude or the structure of the solution [14]. Antone and Al-Maaitah [15] expanded the classes of the solvable linear second-order differential equations with variable coefficients but they did not investigate problems with transient characteristics similar to the one considered in this work.

To demonstrate the dangers of such mistake in dealing with transient vibration problem a linear parametrically excited model equation with variable frequency is considered in the present work. The exact analytical solution of this model equation is then presented. Although the steady state behavior of the frequency converges to a periodical form, which contains no distinguishable transient characteristics, the influence of these characteristics has drastic influence on the steady state solution regardless of the delay in the switch-on time.

MATHEMATICAL MODEL

The general form of linear differential equations that describe non-damped self-excited systems is given by:

$$\ddot{y} + \omega(t) y = 0 ,$$

where $\omega(t)$ is the time varying frequency. For the basic Mathieu's equation $\omega(t)$ is strictly periodic.

Nonetheless, in many engineering applications $\omega(t)$ has transient features before it symptomatically converges to a periodic form. In the present work the model described by the following equation is considered:

$$\frac{d^2 y}{dt^2} + \left\{ \cos t - \sin^2 t + \frac{2m \sin t}{t} - \frac{m(m-1)}{t^2} \right\} y = 0 , \qquad (1)$$

where y and $d^2 y / dt^2$ correspond to the non-dimensional displacement and the non-dimensional acceleration respectively, m is a constant parameter representing certain characteristics of the transient behavior in frequency modulation, and t is the time.

It is clear that eqn. (1) is non-singular for $t>0$. In the present model, the frequency modulation has periodic components as represented by the first two terms between the curly brackets and transient decaying components as represented by the third and fourth terms between the curly brackets. The third term decays in a periodic fashion while the fourth term decays continuously.

The parameter m affects the rabidity of decaying of transient frequency components. It also affects the amplitude of the initial shock in the frequency modulation. For m=0 the frequency is always periodic. Figure 1 shows the variation of $\omega(t)$ with time for different values of m. To demonstrate the convergence of the frequency to a periodic form fig. 2 shows the Poincaret map of the frequency modulation for positive and negative values of m. It is clear

that the steady state form of the frequency converges to a close cycle periodic form. One should keep that in mind while investigating the solution of the model equation.

THE ANALYTICAL SOLUTION

Antone and Al-Maaitah [15] introduced many classes of solvable linear second order differential equation with variable frequency. However, the model equation considered in the present work is not directly related to their work or any previous work in the area (see Takoyama [14]). Nonetheless, the solution procedure presented here for the model equation (1) is inspired from the methods proposed in such works. As such, consider the following transformation

$$y = \Psi e^{\cos t}. \tag{2}$$

Then

$$\frac{dy}{dt} = \frac{d\Psi}{dt} e^{\cos t} - \sin t \Psi e^{\cos t} \tag{3}$$

and

$$\frac{d^2 y}{dt^2} = \left\{ \frac{d^2\Psi}{dt^2} - 2\sin t \frac{d\Psi}{dt} - \cos t \frac{d\Psi}{dt} + \Psi \sin^2 t \right\} e^{\cos t}. \tag{4}$$

Substituting equations (2) and (4) into equation (1), rearranging, and factoring out the term $e^{\cos t}$ we obtain the following equation:

$$\frac{d^2\Psi}{dt^2} - \sin t \frac{d\Psi}{dt} + \left\{ \frac{2m}{t} \sin t - \frac{m(m-1)}{t^2} \right\} \Psi = 0. \tag{5}$$

Equating the coefficients of $(\sin t)^1$ and $(\sin t)^0$ to zero, the following equations can be obtained

$$\frac{d\Psi}{dt} - \frac{m}{t}\Psi = 0 \tag{6}$$

and

$$\frac{d^2\Psi}{dt^2} - \frac{m(m-1)}{t^2}\Psi = 0. \tag{7}$$

A solution to both equations is

$$\Psi = t^m. \tag{8}$$

Consequently, equation (8) is a solution of equation (5). This would finally lead to a solution of equation (1) as follows

$$y_1 = t^m e^{\cos t}. \tag{9}$$

However the general solution of (1) is in the form

$$y = C_1 y_1 + C_2 y_2 , \tag{10}$$

where C_1 and C_2 are arbitrary constants, y_2 is a second solution which is linearly independent of y_1. In general y_2 can be constructed as follows(see reference [16])

$$y_2 = y_1 \int_{t_0}^{t_1} \frac{dt}{y_1^2} .$$

Hence

$$y_2 = t^m e^{\cos t} \int_{t_0}^{t_1} t^{-2m} e^{-2\cos t} dt , \tag{11}$$

where t_0>0 is the initial time at which the system is switched on (the time when initial conditions are applied).

The initial conditions of equation (1) are:

$$y(t_0) = y_0 , \tag{12}$$

$$\frac{dy}{dt}(t_0) = \dot{y}_0 = v_0 , \tag{13}$$

where y_0 and v_0 are the initial displacement and velocity, respectively.

From equations (9), (10), (12), and (13) the following expression for the constants C_1 and C_2 can be derived

$$C_1 = \frac{y_0}{t_0^m e^{\cos t_0}} , \tag{14}$$

$$C_2 = \dot{y}_0 t_0^m e^{\cos t_0} - y_0 (m t_0^{m-1} - \sin t_0^m) e^{\cos t_0} . \tag{15}$$

Of course, either C_1 or C_2 can be set to zero by choosing the appropriate initial conditions. It should be also noted that although y_1 is defined in an analytical close from y_2 is not. Nonetheless, the integral in y_2 can be evaluated numerically with sufficient accuracy to be considered as quasi-analytical. The physical characteristics of the solution to equation (1) are discussed below.

RESULTS AND DISCUSSION

The model equation (1) was chosen for two main reasons. The first is that it simulates certain cases of engineering application with many interesting features demonstrating the drastic effect of transient frequency modulation on the steady state response of the system. The second reason is that an analytical solution of that model can be found leading to clear and easy investigation of its characteristics.

As it is discussed earlier the transient features of frequency modulation decay completely after a certain period of time depending on the value of *m*. It is demonstrated that transient

behavior of ω is indistinguishable as t>20 for the cases of $m = -0.1$, 0, and +0.1 (please refer to figures 1 and 2). However, the transient behavior of ω results in a drastic effect on the system's response as the analytical solution of equation (1) illustrates.

Equation (9) clearly demonstrate that the first component of the solution (y_1) is periodically growing when m>0 and periodically decaying when m<0, while it is strictly periodic when m=0. Figures 3 and 4 show the behavior of y_1 for various values of negative and positive m. When m=0 the first component of the solution y_1 has a limit cycle of an amplitude around 2.75. On the other hand y_1 reaches a value of $8{\cdot}10^{11}$ at $t \approx 200$ when m=5. For negative values of m the first component of the solution decays with time in a rate depending on the value of m as shown in figures 4.b–4.c.

It should be noted that although the discussion above is only for one component of the solution it could be the sole component of the general solution of the model if the right initial conditions are chosen. From equations (10) and (15) the general solution becomes equal to ($C_1 y_1$) if the initial velocity is chosen to be

$$\dot{y}_0 = y_0\left(\frac{m}{t_0} - \sin t_0\right). \tag{16}$$

In other words, the initial velocity is chosen such that C_2 in equation (10) is identically zero. In this case, the transient effect of the frequency modulation can directly lead to bounded or unbounded response of linear self-excited systems depending on the sign of the component m. It is worthwhile to remember that the effect of m on the frequency modulation vanishes with time.

In general, however, there are tow components of the solution to equation (1). The second solution to this equation, y_2, can be found using equation (11) noting that the integration in this equation can be done numerically with great accuracy. Although y_1 is bounded when m=0, figure 5.a shows that y_2 is growing since the value in the integrand is always positive. Figures 5.a–5.d show the variation of y_2 with time in for different values of positive m. At $t \approx 200$ the amplitude of y_2 reached $12{\cdot}10^{18}$ when m=5 while it only reached 900 when m=0.1 at the same time. For negative values of m, on the other hand, figures 6.a–6.d show y_2 as function of time.

Although the general solution is always unbounded if $C_2 = 0$, the effect of m on the structure and the (order of) magnitude of that general solution is drastic. This is quite important especially since such growth might have dangerous effect in triggering non-linearity in the physical problem. As such, samples of the total general solution of equation (1) are presented by choosing arbitrary initial conditions that do not quench any of the two components of the general solution. Here, the initial conditions are chosen to be

$$\dot{y}_0(0.1) = y_0(0.1) = 0.1.$$

Both C_1 and C_2 are then calculated and the general solution is then found from equation (10). For m=0 the general solution is always growing as shown in figure 7 and its amplitude reaches 350 at $t \approx 200$. For m=0.1. However, the amplitude of the total solution is much less than that when m=0 as shown in figure 8.a. For $m = -0.1$ the amplitude of the solution reaches to 1200 at $t \approx 200$ as shown in figure 8.b while figure 9 shows a comparison of the general solution of y for $m = \pm 1.0$. It can be noted that not only the amplitude of the solution is at a much lower scale of amplitude for $m = +0.1$ as compared to that when $m = -0.1$, but also y is always negative when $m = +1.0$ while it is always positive when $m = -1.0$. This means that the transient characteristics of the frequency modulation affect the structure of the response as well as its amplitude. For the cases of $m = \pm 5.0$ the general solution is shown in figure 10. The drastic effect of the transient behavior of the frequency variation on the response of the modelled self-excited system is clearly demonstrated in the discussion above.

NUMERICAL SOLUTION

The full numerical solution of equation (1) is presented here for two reasons. The first is to compare the numerical solution with the quasi-analytical solution derived in this work. The second reason is to demonstrate the dangers of neglecting the transient behavior of the frequency modulation on the steady state solution of parametrically excited systems. Equation (1) is then solved using a fourth-order Runge-Kutta technique with a step size in time of 0.01. Figure 11 shows the numerical solution of equation (1) when m=0 and the same initial conditions of figure 7. The numerical solution completely matches the analytical solution. In fact all the analytical solutions found in figures 8 and 9 were also reproduced numerically with perfect match.

An interesting case is that when the initial conditions are chosen to eliminate the growing solution y_2 from the general solution. When $m = -0.1$ the analytical solution is only a decaying one. However, the accumulation of the numerical error in the numerical solution combined with the numerical inaccuracy of the initial conditions would trigger y_2 to grow again in the numerical solution. This is demonstrated in figure 12, which shows the decaying of the response for a certain period of time after which the numerical response starts to grow again. The initial conditions in figure 12 are applied at (t=100). At this time the transient modulation of the frequency can be considered as non-existing. For these conditions one might mistakenly think that equation (1) can be approximated by the following equation after neglecting the transient components of the frequency variation

$$\frac{d^2 y}{dt^2} + \{\cos t - \sin^2 t\} y = 0 . \tag{17}$$

The analytical solution of eqn. (17) can be found by setting m=0 while the general analytical solution of equation (1) under these initial conditions is simply ($C_1 e^{\cos t} / t^{0.1}$). As can be shown from the discussion above these two solutions are completely different.

CONCLUSION

A certain model equation of linear parametrically excited system is chosen and the analytical solution of this equation is derived for the first time. Both the analytical and the numerical solutions of the chosen model equation demonstrate the drastic effect of the transient frequency modulation on the long term solution of such systems. It is misleading to assume that if some characteristics of the frequency variation vanish with time then the effect of these characteristics on the system's response can be neglected. In other words, the transient characteristics of frequency modulation have a non-transient effect that might drastically affect the response both the structure and the amplitude of the solution.

REFERENCES

1. Jaswal, J .S. and Bhave S. K., Experimental evaluation of damping in a bladed disk model. J. of Sound and Vibration, 177(4), 111–120, (1994).
2. Bently, D. E. and Muszynska, A. Perturbation study of a rotor/bearing system: identification of the oil whip resonance. ASME Design Engineering Division Conference and Exhibit on Mechanical Vibration and Noise, Cincinnati, Ohio, September 10–13, 189–198, (1985).
3. Kirk, R. G. and Gunter, E. J. Transient response of Rotor-Bearing Systems. ASME J. of Engineering for Industry, 81(2), 682–693, (1974).
4. Cveticanin, L. Self-Excited vibrations of the variable mass rotor/fluid system. J. of Sound and Vibration, 212(4), 685–702, (1998).
5. Alam, M. and Nelson, H. D. A blade loss response spectrum for flexible rotor systems. Transaction of the ASME J. of Engineering for Power, Paper No. 84-GT-29. (1983).
6. Nayfeh, A. H. and Mook, D. T. *Nonlinear Oscillation.* John Wiley and Sons, New York, (1979).
7. Mitropoloskii, Yu. A., and Van Dao, N. *Applied Asymptotic Method in Non-linear Oscillations.* Kluwer, Dordrecht, (1997).
8. Sanliturk, K. Y. and Ewins, D. J. Modeling two-dimensional friction contact and its application using harmonic balance method. 193(4), 511–523, (1996).
9. Csaba, C. and Anderson, M. Optimization of friction damper weight, simulation and experiments. ASME Turbo Expo 97, Orlando, FL, USA, paper No. 97-GT-115, (1997).
10. Zienkewicz, O. C. *The Finite Element Method.* 4th ed. McGraw-Hill Book Company, London, (1991).
11. Irretier, H. and Balashov, D. B. Transient response oscillations of a slow-variant systems with small non-linear damping: Modelling and prediction. J. of Sound and Vibration, 231(5), 1271–1287, (2000).
12. Irretier, H. and Leul, F. Non-stationary vibrations of mechanical systems with slowly varying natural frequencies during acceleration through resonance. Proceedings of the 9th World Congress on the Theory of Machine and Mechanisms, 1319–1323, (1995).
13. Asfar, K. R. Quenching of self-excited vibration. J. of Vibration, Acoustics, Stress, and Reliability in Design, 111(2), 130–133, (1989).
14. Takayama, K. A. Class of solvable second order ordinary differential equations with variable coefficients. J. of Mathematical Physics, 27(4), 1747–1762, (1984).
15. Antone, T. A., Al-Maaitah, A. A. Analytical solutions to classes of linear oscillator equations with time varying frequencies. J. Mathematical Physics, 33(10), 3330–3339, (1992).
16. Coddington, E. A. and Levinson, N. *Theory of Ordinary Differential Equations.* McGraw-Hill, New York, (1955).

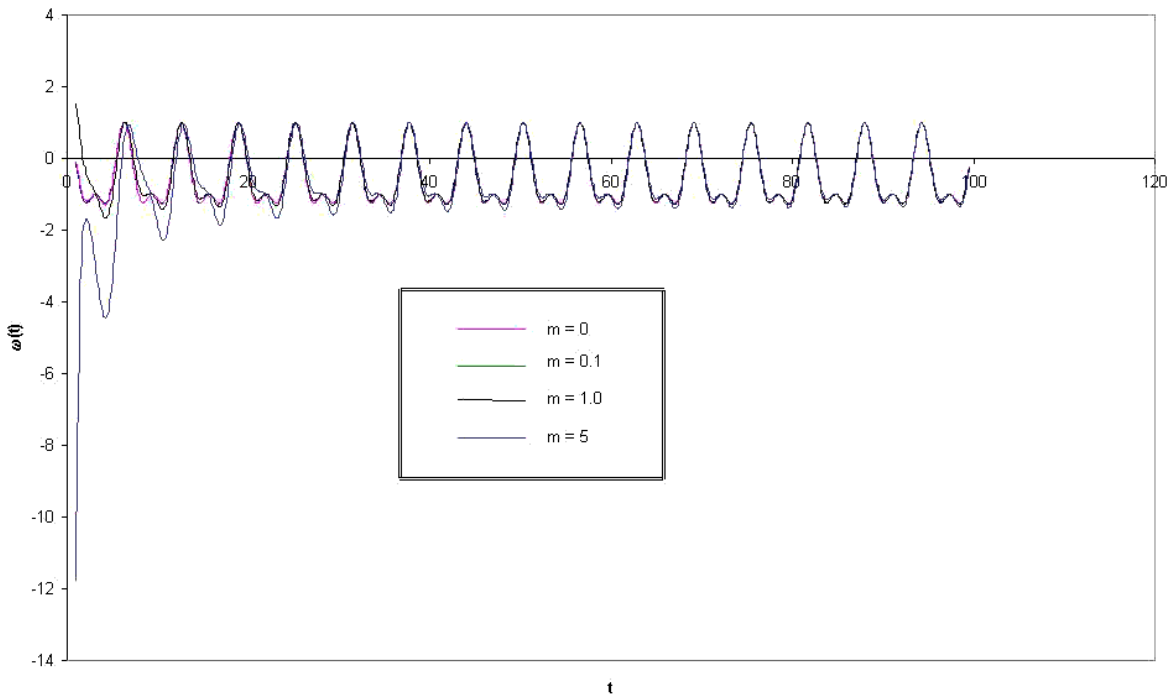

Fig. 1. Time varying frequency for different values of *m*

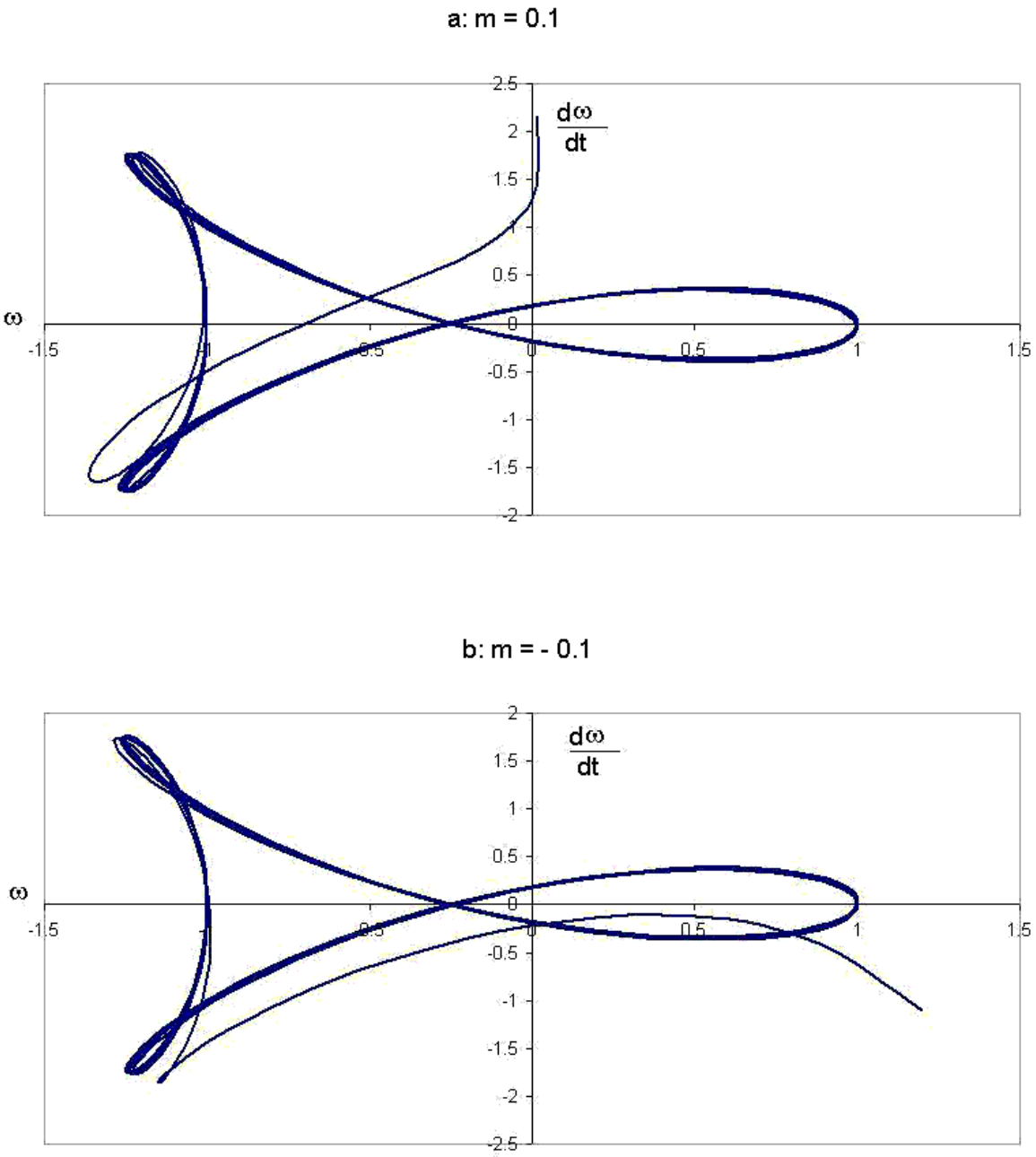

Fig. 2. Poincaret map of frequency modulation for m=± 0.1

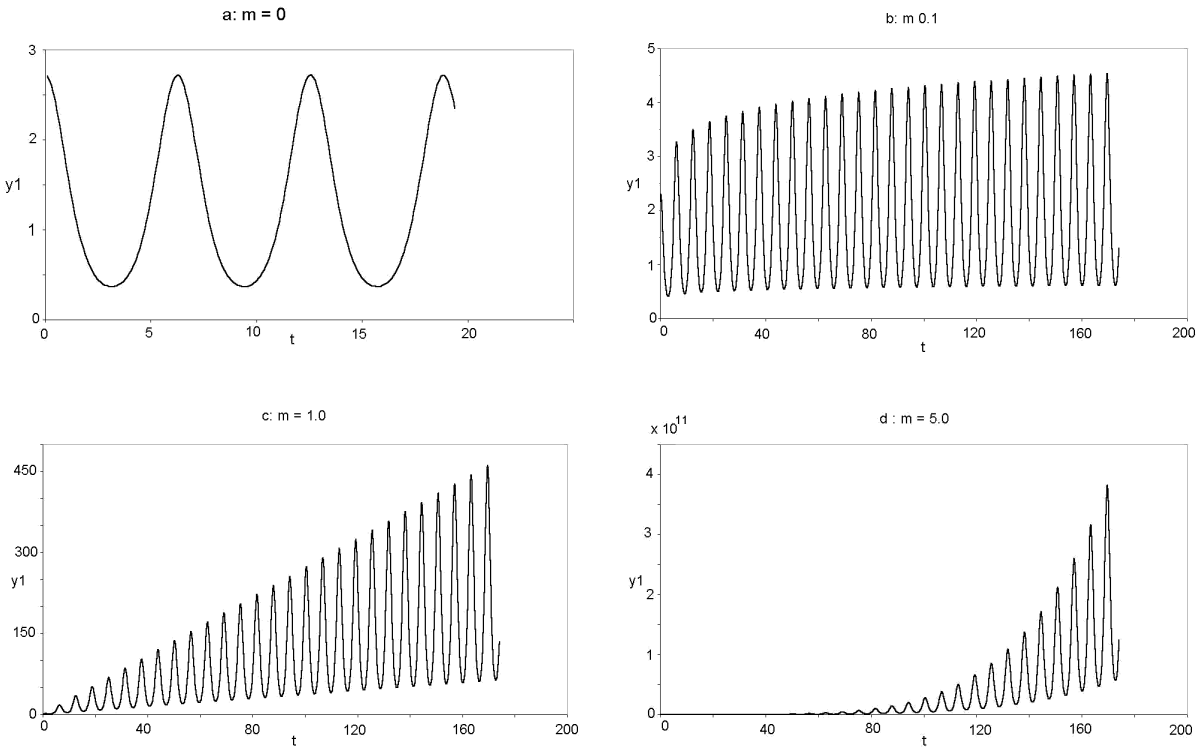

Fig. 3. y_1 for various m

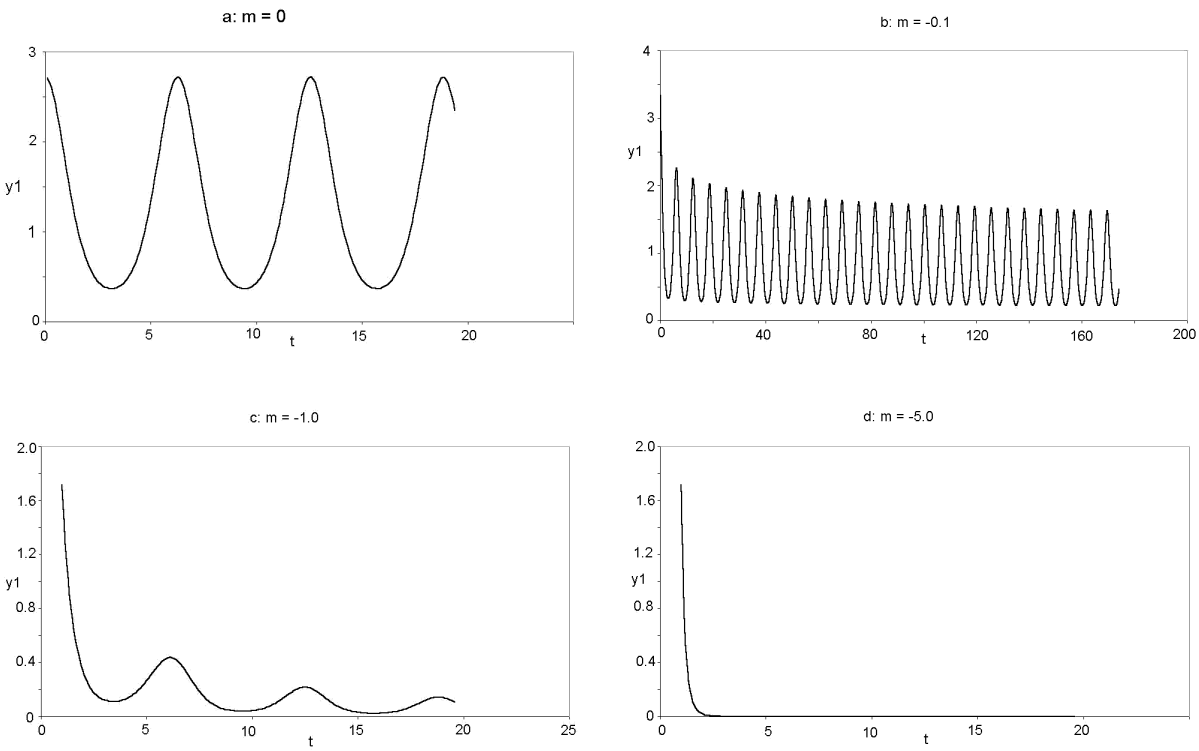

Fig. 4. y_1 for various m

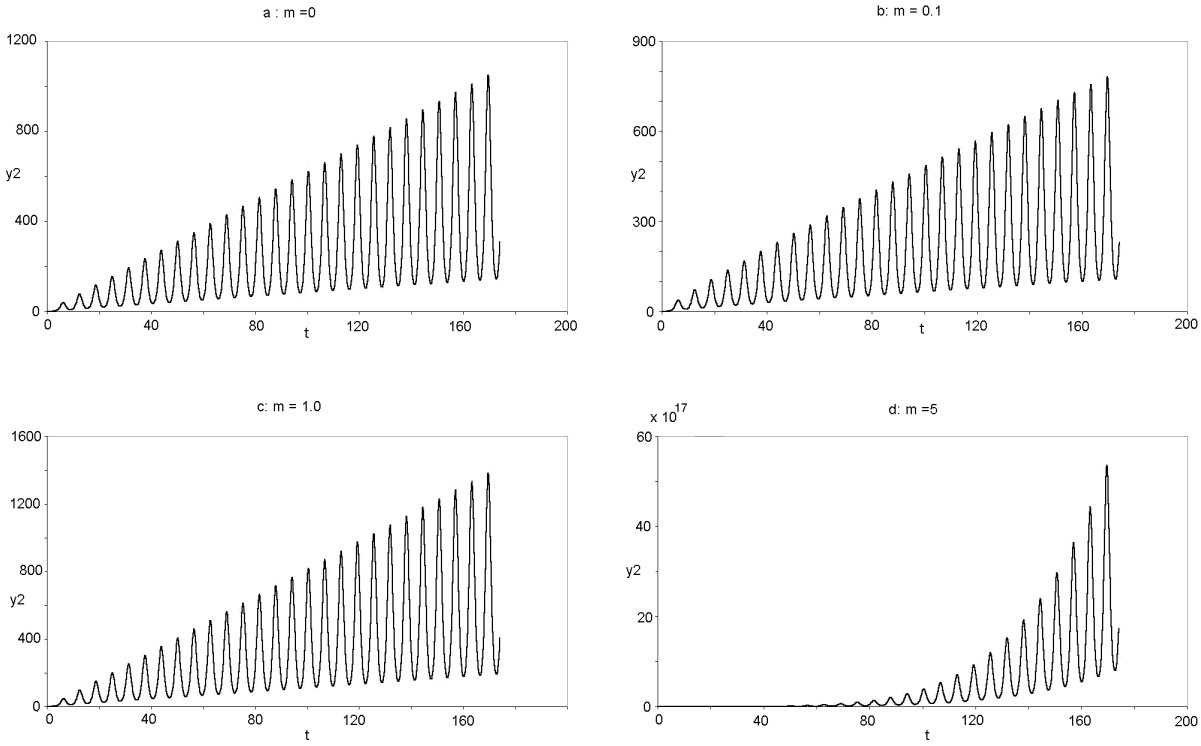

Fig. 5. y_2 for various m

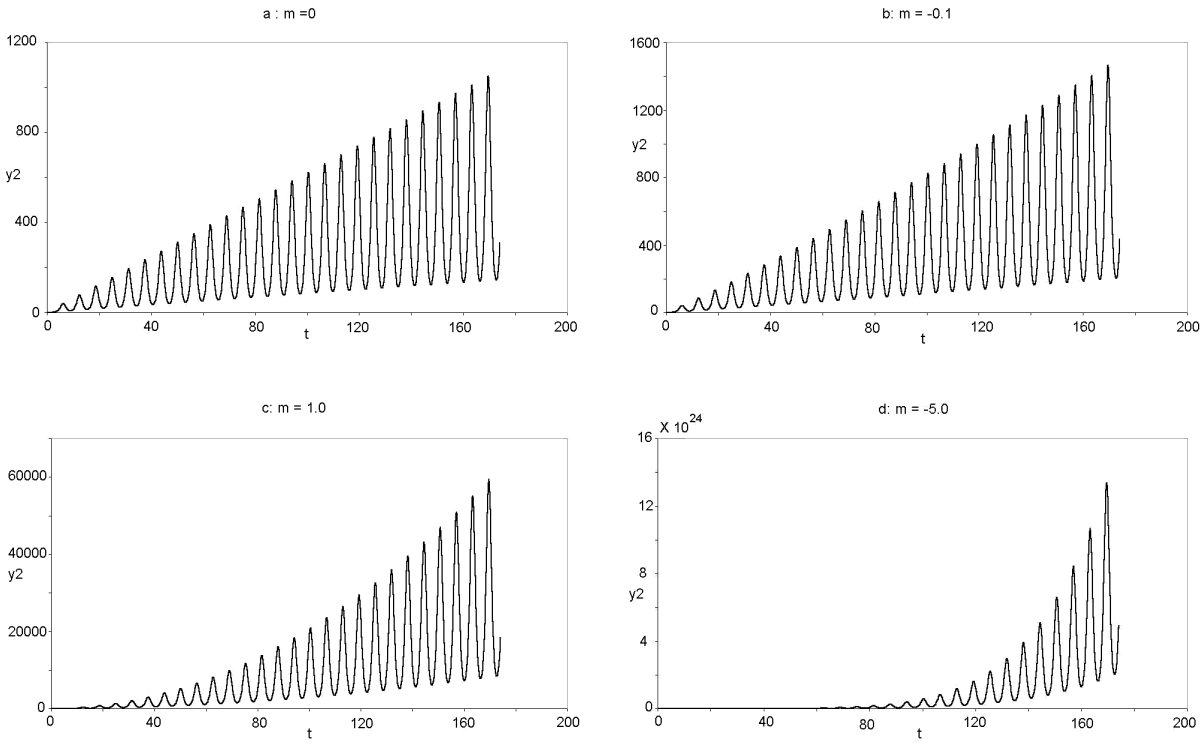

Fig. 6. y_2 for various m

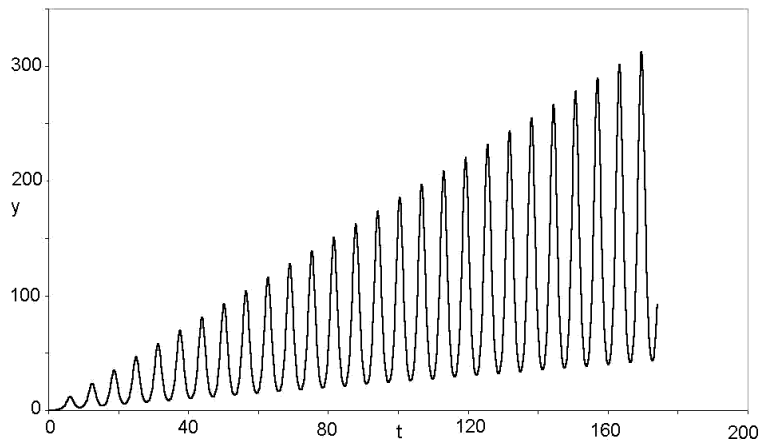

Fig. 7. The total analytical solution for $m = 0$ and the initial conditions $y_0(0.1) = y_0'(0.1) = 0.1$

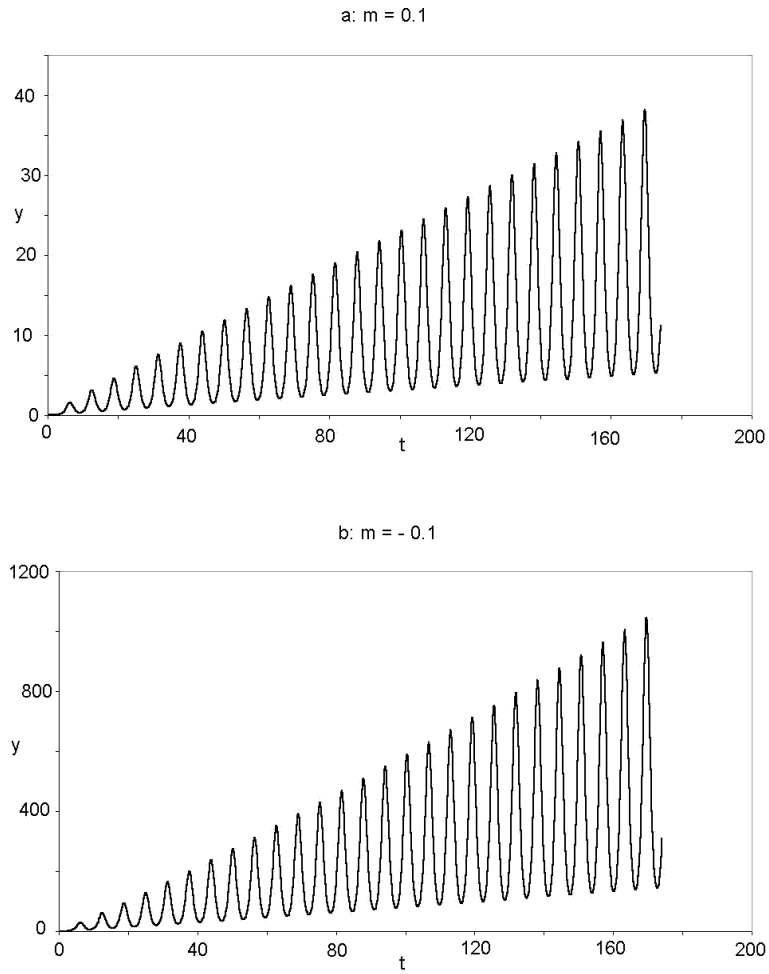

Fig. 8. The total analytical solution for $m = \pm 0.1$ and the initial conditions $y_0(0.1) = y_0'(0.1) = 0.1$

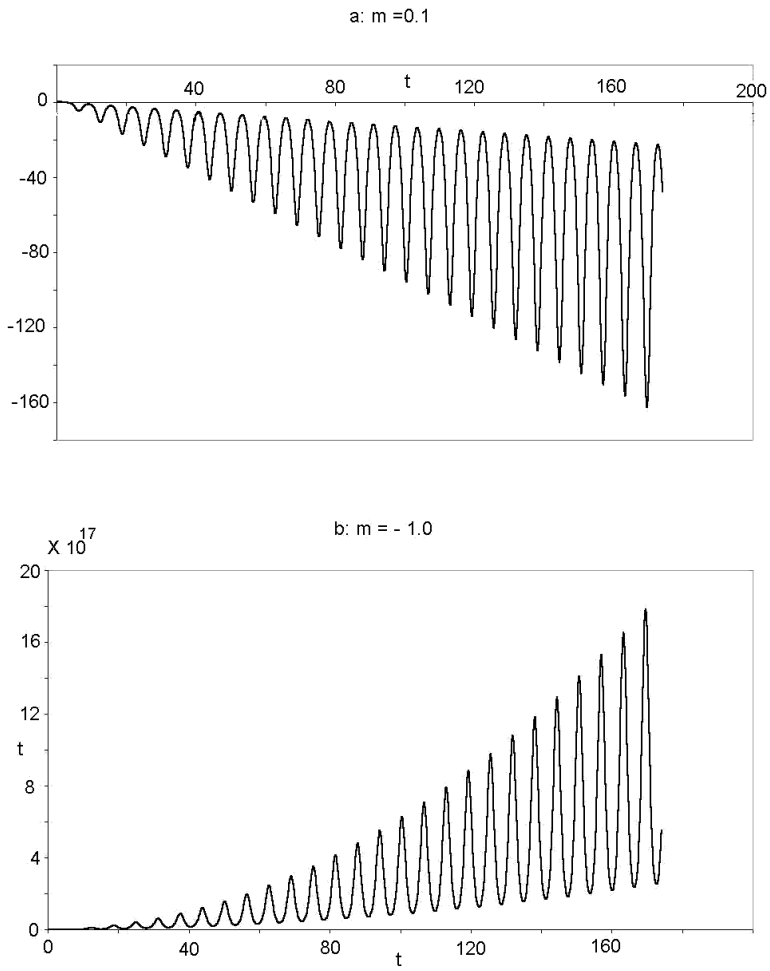

Fig. 9. The total analytical solution for $m = \pm 1.0$ and the initial conditions $y_0(0.1) = y_0'(0.1) = 0.1$

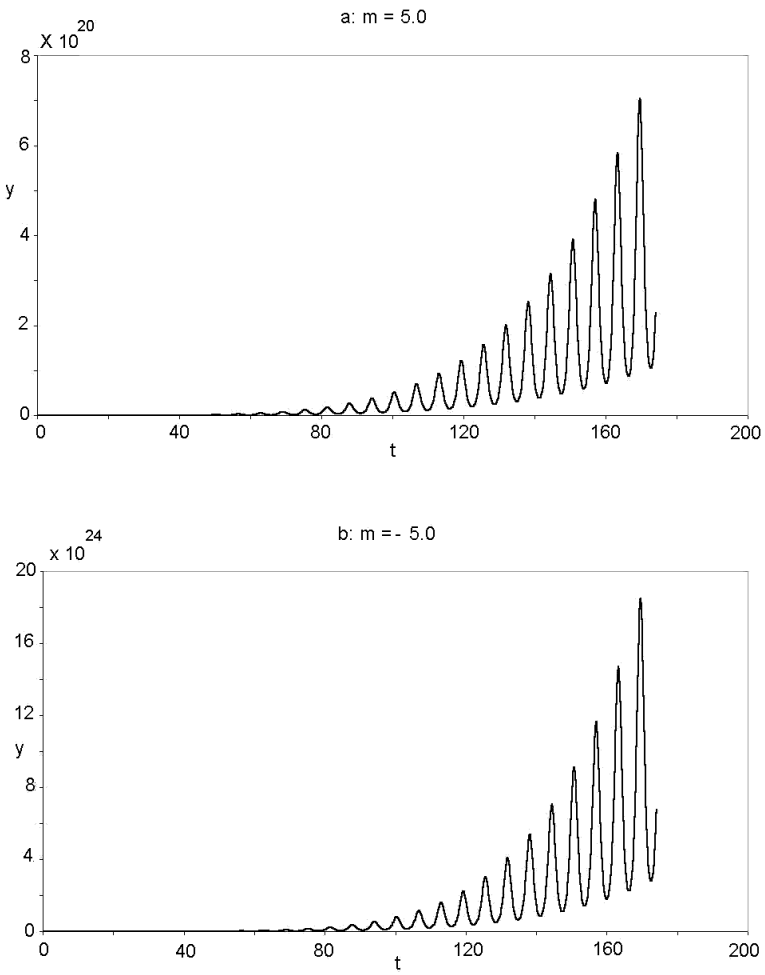

Fig. 10. The total analytical solution for $m = \pm 5.0$ and the initial conditions $y_0(0.1) = y_0'(0.1) = 0.1$

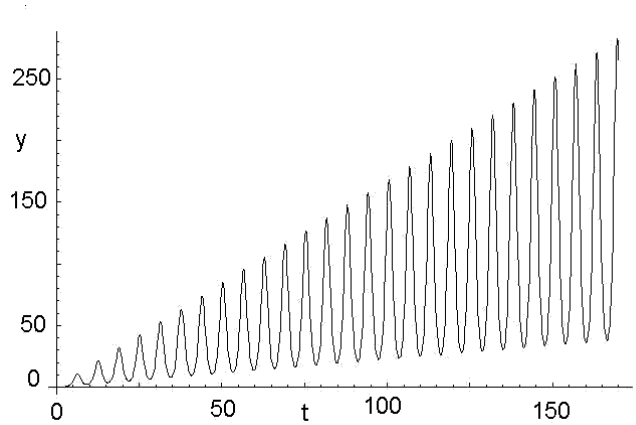

Fig. 11. The numerical solution for $m = 0$ and the initial conditions $y_0(0.1) = y_0'(0.1) = 0.1$ using a highly accurate numerical schemes

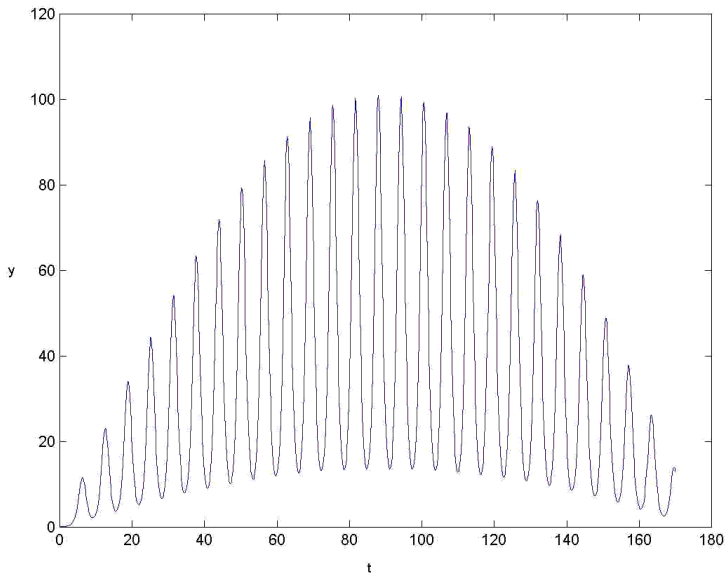

Fig. 12. The numerical solution for $m = 0$ and the initial conditions $y_0(0.1) = y_0'(0.1) = 0.1$ using fourth-order Runge-Kutta technique in MATLAB