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An analytic solution of the stagnation point flow problem

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The Falkner-Skan equation models the laminar flow of an incompressible fluid for several physical situations. A specially interesting case is that of a flow with a stagnation point. This problem is solved analytically in the form of a power series with a finite radius of convergence. By modifying a Pade approximant corresponding to the power series a simple expression is obtained which describes the solution uniformly over the whole domain $[0,\infty)$.

Keywords: Falkner-Skan equation, stagnation point flow, Pade approximation, Wang transformation

INTRODUCTION

The Falkner-Skan problem

$$f'''(\eta)+\beta_0 f(\eta) f''(\eta)+\beta[1-[f'(\eta)]^2]=0,\ \ \eta\ge 0, \tag{1.1a}$$

$$f(0)=0,\ f'(0)=0, \lim_{\eta\to\infty} f'(\eta)=1 \tag{1.1b}$$

describes the laminar flow of an incompressible fluid under a variety of circumstances.

When $\beta_0=1$, the above problem corresponds to decelerating, constant and accelerating flows for respectively $\beta<0, \beta=0$ and $\beta>1$. If $\beta_0=\frac{1}{2}$ and $\beta=0$, we get the Blasius problem

$$f'''(\eta)+\frac{1}{2} f(\eta) f''(\eta)=0,\ \ \eta\ge 0 \tag{1.2}$$

with the boundary conditions (1.1b).

If $\beta_0=1$ and $\beta=1$ then the equation

$$f'''(\eta)+f(\eta) f''(\eta)+1-[f'(\eta)]^2=0,\ \eta\ge 0 \tag{1.3}$$

with boundary conditions (1.1b) represents the stagnation point flow [1].

The above problems are two-point nonlinear boundary value problems and it is difficult to find analytical solutions which are valid outside a finite interval. A numerical solution of (1.2) was first obtained by Howarth [2] and the Falkner-Skan problem was first solved numerically

by Hartree [3]. Subsequently several authors have devised numerical techniques to efficiently deal with the two problems, see Asaithambi [4, 5] and references therein.

Blasius [6] found an analytical solution of (1.2) in the form of a power series, however the series converges only for $0 \le \eta \le 0.5690$. Liao [7], by using his homotopy analysis method, found an analytical solution valid on the whole domain $[0,\infty)$. A power series solution of (1.3) may be found by using, for example, Adomian decomposition method, but the series converges only in $[0,\eta_0)$ where $\eta_0 \approx 3.2$ [8]. To our knowledge no analytical solution of (1.3) exists which represents the function on the whole domain $[0,\infty)$. In this paper we shall present a simple expression which fits the function $f'(\eta)$ on $[0,\infty)$ and a simple quadrature gives $f(\eta)$ on the whole domain with remarkable accuracy.

1. ANALYTICAL SOLUTION OF FALKNER-SKAN EQUATION

Asghar [8] found a power series solution by using the Adomian decomposition method. Here we find this solution by a direct method which produces the coefficients in the series with less effort.

Denote the unknown $f''(0)$ by α and let a solution of (1.3) be

$$f(\eta) = \sum_{n=0}^{\infty} a_n \eta^n . \tag{2.1}$$

Substitution of f, f'', f''' in (1.3) yields

$$\begin{aligned} &\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}\eta^n + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}a_k\eta^{n+k} + 1 \\ &- \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(k+1)(n+1)a_{k+1}a_{n+1}\eta^{n+k} = 0 . \end{aligned} \tag{2.2}$$

Consider the second summation in (2.2) i.e.

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}a_k\eta^{n+k} \tag{2.3}$$

and define a new index $m = n + k$, hence $n = m - k$, which indicates $0 \le n \le m$. With the new indices n and m the summation in (2.3) becomes

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}(n+2)(n+1)a_{n+2}a_{m-n}\eta^m . \tag{2.4}$$

Similarly the last summation in (2.2) can be replaced by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}(m-n+1)(n+1)a_{m-n+1}a_{n+1}\eta^m . \tag{2.5}$$

Thus (2.2) can be written as

$$\sum_{m=0}^{\infty}(m+3)(m+2)(m+1)a_{m+3}\eta^m + \sum_{m=0}^{\infty}\sum_{n=0}^{m}(n+2)(n+1)a_{n+2}a_{m-n}\eta^m + 1$$
$$-\sum_{m=0}^{\infty}\sum_{n=0}^{m}(m-n+1)(n+1)a_{m-n+1}a_{n+1}\eta^m = 0. \tag{2.6}$$

From the boundary conditions (1.1b) we have

$$a_0 = 0, \qquad a_1 = 0. \tag{2.7}$$

While the assumption $f''(0) = \alpha$ implies

$$a_2 = \frac{\alpha}{2}. \tag{2.8}$$

By equating the constant term in (2.6) to zero we get

$$a_3 = -\frac{1}{6}. \tag{2.9}$$

Higher coefficients a_m, $m \geq 1$ can be found from the following recurrence relation which we get by setting the coefficient of η^m in (2.6) to zero

$$a_{m+3} = -\frac{1}{(m+3)(m+2)(m+1)}[\sum_{n=0}^{m}(n+2)(n+1)a_{n+2}a_{m-n}$$
$$-(m-n+1)(n+1)a_{m-n+1}a_{n+1}], \quad m \geq 1. \tag{2.10}$$

First few coefficients given by (2.10) are as follows

$$a_4 = 0, a_5 = \frac{\alpha^2}{120}, a_6 = -\frac{\alpha}{360}, a_7 = \frac{1}{2520},$$
$$a_8 = \frac{\alpha^3}{40320}, a_9 = \frac{\alpha^2}{90720}, a_{10} = -\frac{\alpha}{226800}, \tag{2.11}$$
$$a_{11} = \frac{16+27\alpha^4}{3991680}, a_{12} = -3.77869\times 10^{-7}\alpha^3.$$

2. WANG EQUATION

By defining

$$x = \frac{df}{d\eta}, \quad y = \frac{d^2 f}{d\eta^2}.$$

Wang [9] transformed equation (1.2) to

$$y\frac{d^2 y}{dx^2} + \frac{1}{2}x = 0.$$

Asghar [8] has shown that the same transformation changes equation (1.3) into

$$y^2 y'' - (1-x^2)y' - xy = 0, \qquad 0 \le x < 1. \tag{3.1}$$

The boundary conditions transform to

$$y(0) = \alpha, y'(0) = -\frac{1}{\alpha}, y(1) = 0. \tag{3.2}$$

Assume a solution of the form

$$y = \sum_{n=0}^{\infty} b_n x^n. \tag{3.3}$$

It is easy to find $b_2 = -\frac{1}{2\alpha^3}$ and proceeding as in section 2, we obtain the recurrence relation

$$b_{m+2} = \frac{1}{(m+1)(m+2)\alpha^2}[-\sum_{n=0}^{m-1}\sum_{l=0}^{m-n}(n+2)(n+1)b_{m-l-n}b_l b_{n+2} + (m+1)b_{m+1} - (m-2)b_{m-1}], \quad m \ge 1. \tag{3.4}$$

First few coefficients generated by (3.4) are as follows

$$b_3 = \frac{0.166667(-3+\alpha^4)}{2\alpha^5}, b_4 = \frac{-0.625 + 0.208333\alpha^4}{\alpha^7}$$

$$b_5 = \frac{-0.875 + 0.333333\alpha^4}{\alpha^9}, b_6 = \frac{-1.3125 + 0.569444\alpha^4 - 0.0222222\alpha^8}{\alpha^{11}}.$$

The unknown parameter α can be found by solving the equation

$$y(1) = 0. \tag{3.5}$$

In practice one finds an approximation α_k to α by replacing (3.5) by

$$\sum_{n=0}^{k} b_n(\alpha) = 0. \tag{3.6}$$

A few members of the sequence $\{\alpha_k\}_{k=0}^{\infty}$ are displayed in Table 1.

Table 1. Approximation values $f''(0) = \alpha$

k	α_k
50	1.2321
90	1.2324
250	1.23254
1000	1.23258

The value of α, correct to seven decimal positions as found numerically by shooting method, is $\alpha = 1.232589$ [5].

3. UNIFORMLY VALID ANALYTICAL SOLUTION

Using the coefficients evaluated in section 2 and the value of α found in section 3 the power series expression for the solution of the stagnation point flow problem up to the power 20 becomes

$$\begin{aligned} f(\eta) = {} & 0.616294\eta^2 - 0.166667\eta^3 + 0.0126606\eta^5 \\ & -0.00342385\eta^6 + 0.000396825\eta^7 - 0.0000464444\eta^8 \\ & +0.0000167468\eta^9 - 5.43469\times10^{-6}\eta^{10} \\ & +1.96211\times10^{-6}\eta^{11} - 7.07612\times10^{-7}\eta^{12} \\ & +2.04944\times10^{-7}\eta^{13} - 6.11228\times10^{-8}\eta^{14} \\ & +2.05581\times10^{-8}\eta^{15} - 6.27563\times10^{-9}\eta^{16} \\ & +1.78703\times10^{-9}\eta^{17} - 5.46273\times10^{-10}\eta^{18} \\ & +1.65946\times10^{-10} - 4.61405\times10^{-6}\eta^{11}\eta^{20}. \end{aligned} \tag{4.1}$$

Differentiate (4.1) to get

$$\begin{aligned} f'(\eta) = {} & 1.23259\eta - 0.5\times10^{-6}\eta^{11} + 0.063303\eta^4 - 0.0205431\eta^5 \\ & +0.00277778\eta^6 - 0.000371555\eta^7 + 0.000150721\eta^8 \\ & -0.0000543469\eta^9 + 0.0000215832\eta^{10} \\ & -8.49134\times10^{-6}\eta^{11} + 2.66427\times10^{-6}\eta^{12} \\ & -8.5572\times10^{-7}\eta^{13} + 3.08372\times10^{-7}\eta^{14} \\ & -1.0041\times10^{-7}\eta^{15} + 3.03796\times10^{-8}\eta^{16} \\ & -9.83291\times10^{-9}\eta^{17} + 3.15297\times10^{-9}\eta^{18} \\ & -9.22811\times10^{-10}\eta^{19}. \end{aligned} \tag{4.2}$$

Applying Pade [6/6] approximation to (4.2) we get

$$f'(\eta) = \frac{f_1(\eta)}{f_2(\eta)}, \tag{4.3}$$

where

$$\begin{aligned} f_1(\eta) = {} & 1.23259\eta + 5.06009\eta^2 + 0.0665943\eta^3 \\ & -0.204148\eta^4 + 0.17892\eta^5 - 0.0139962\eta^6 \end{aligned}$$

and

$$\begin{aligned} f_2(\eta) = {} & 1 + 4.51091\eta + 1.88388\eta^2 + 0.547214\eta^3 \\ & +0.152132\eta^4 + 0.0265334\eta^5 + 0.00419318\eta^6. \end{aligned}$$

Now we replace the functions $f_1(\eta)$ and $f_2(\eta)$ by respectively $g_1(\eta)$ and $g_2(\eta)$ so that

$$f'(\eta) = \frac{g_1(\eta)}{g_2(\eta)}, \tag{4.4}$$

where

$$g_1(\eta) = 1.23259\eta + 5.06009\eta^2 + 0.0665943\eta^3 - 0.204148\eta^4$$
$$+0.17892\eta^5 - 0.0139962\eta^6 + a\eta^7\left(\frac{e^{\eta 2}}{2} - 1\right)$$

and

$$g_2(\eta) = 1 + 4.51091\eta + 1.88388\eta^2 + 0.547214\eta^3 + 0.152132\eta^4$$
$$+0.0265334\eta^5 + 0.00419318\eta^6 + a\eta^7\left(\frac{e^{\eta 2}}{2} - 1\right),$$

where a is a constant to be determined. It is clear that for every a the expression (4.4) will represent $f'(\eta)$, for small η, since it will differ only slightly from the $[6/6]$ Pade approximant for the function. Also for large η the exponential term will dominate other terms and the limiting value will approach unity as required by the boundary condition $f'(\infty) = 1$. An optimum choice of a should ensure that the ratio $g_1(\eta)/g_2(\eta)$ differs as little as possible from the exact value of $f'(\eta)$ on the entire domain $[0,\infty)$. To achieve this we numerically evaluate the integral

$$\int_0^{\eta_\infty} \frac{g_1(\eta)}{g_2(\eta)} d\eta \tag{4.5}$$

and choose a so as to make the above integral as close to $f(\eta_\infty)$ for some suitable but arbitrary η_∞. We choose $\eta_\infty = 8$ which forces us to fix $a = 0.00051$. With this choice of parameters the integral (4.5) gives 7.35212 which agrees with the exact numerical value to four decimal positions. Also

$$\frac{g_1(8)}{g_2(8)} = 1,$$

which implies that the function $g_1(\eta)/g_2(\eta)$ has already attained its asymptotic behavior at 8 and any integration beyond this point will yield approximate values for $f(\eta)$ in agreement with the exact values to at least four decimal positions. Let $f_a(\eta)$ denote an approximate value found from

$$f_a(\eta) = \int_0^{\eta} \frac{g_1(u)}{g_2(u)} du\,. \tag{4.6}$$

In Table 2 we compare a few approximate values, as found from (4.6), with the exact values of the solution, obtained numerically.

Table 2. Comparison of approximate values with exact

η	Approximate $f_a(\eta)$ from (4.6)	$f(\eta)$ from a numerical solution
0	0	0
0.4	0.0880566	0.0880566
0.8	0.312428	0.312423
1.2	0.622033	0.622028
1.6	0.9798	0.9798
2	1.3620	1.3621
2.4	1.7555	1.7552
2.8	2.1534	2.1530
3.2	2.5526	2.5523
3.6	2.9523	2.9522
4	3.3522	3.3521
4.4	3.7521	3.7521
4.8	4.1521	4.1521
5.2	4.5521	4.5521
5.6	4.9521	4.9521
6	5.3521	5.3521
6.4	5.7521	5.7521
6.8	6.1521	6.1521

It is clear that for $0 \le \eta < \infty$, the maximum error is less than 2 parts in ten thousand and it steadily decreases as we go down the Table. Even this error could be decreased by considering a Pade approximant of an order higher than $[6/6]$ considered in this paper.

CONCLUSIONS

Sometimes it is possible to find an approximate solution of a nonlinear problem in the form of an infinite series. However the series may not converge outside a finite interval whereas the solution is known to exist on an infinite domain. In the case of the Blasius problem, the domain was extended to infinity by Liao [7] by making use of the homotopy analysis method and by Ahmad and Albarakati [10] by employing a technique similar to the one used in the present work. It is possible to use the homotopy technique to solve Falkner-Skan equation, however this will result in lengthy expressions. We have been successful in finding a *short* expression because, by modifying the Pade approximation, we have incorporated the asymptotic behavior of the function. As a result this expression gains in accuracy as the values of η are increased in contradistinction to solutions in the form of series or rational expressions.

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